

# EXAMPLES OF AUSLANDER-REITEN COMPONENTS IN THE BOUNDED DERIVED CATEGORY

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**ABSTRACT.** We deduce a necessary condition for Auslander-Reiten components of the bounded derived category of a finite dimensional algebra to have Euclidean tree class by classifying certain types of irreducible maps in the category of complexes. This result shows that there are only finitely many Auslander-Reiten components with Euclidean tree class up to shift. Also the Auslander-Reiten quiver of certain classes of Nakayama are computed directly and it is shown that they are piecewise hereditary. Finally we state a condition for  $\mathbb{Z}[A_\infty]$ -components to appear in the Auslander-Reiten quiver generalizing a result in [W].

## 1. INTRODUCTION

Auslander-Reiten triangles have been introduced by Happel for triangulated categories generalizing the concept of Auslander-Reiten sequences for finite-dimensional algebras.

In this paper we analyze Auslander-Reiten triangles in the bounded derived category of a finite-dimensional algebra  $A$ , denoted by  $D^b(A)$ . Using Auslander-Reiten triangles, Happel has defined a locally finite graph, whose vertices correspond to indecomposable complexes in  $D^b(A)$  and whose edges correspond to irreducible maps appearing in Auslander-Reiten triangles. The connected components of this quiver are the Auslander-Reiten components.

So far Auslander-Reiten components of  $D^b(A)$  for  $A$  self-injective have been determined by Wheeler in [W] and for  $A$  a hereditary algebra they have been determined by Happel in [H1, I,5.5]. But very little is known about the components of the bounded derived category of non self-injective algebras. In this paper we compute the Auslander-Reiten quiver of the bounded derived category for a class of algebras with finite global dimension.

In [S] we classified bounded derived categories whose Auslander-Reiten quiver has either a stable component of Dynkin tree class, or a stable finite component or a bounded component. It is shown that a component with

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Dynkin tree class can only appear in the Auslander-Reiten component of the bounded derived category of a piecewise hereditary algebra. In this paper which can be read independently from [S] we will focus on Auslander-Reiten components with Euclidean tree class.

In the first section we introduce some notation and give the definition of Auslander-Reiten triangles. In section two we state some properties of Auslander-Reiten triangles and show that certain indecomposable elements lie in Auslander-Reiten components isomorphic to  $\mathbb{Z}[A_\infty]$ . This generalizes Wheeler's main result in [W].

In section three we determine a necessary condition for the existence of stable Auslander-Reiten components with Euclidean tree class. In the last section we determine the Auslander-Reiten quivers of the bounded derived category for certain classes of algebras and use our results to show that they are piecewise hereditary.

In the third section irreducible maps of complexes in  $Comp^{-,b}(\mathcal{P})$  that end in an indecomposable contractible complex are analyzed. We show that these irreducible maps start in a complex that is isomorphic in  $D^b(A)$  to a simple module of  $A$  embedded into  $D^b(A)$ . Using this result we show that a stable Auslander-Reiten component of  $D^b(A)$  which has Euclidean tree class contains a simple  $A$ -module. It then follows that there are only finitely many Auslander-Reiten components with Euclidean tree class up to shift and this number is bounded by the number of isomorphism classes of simple modules.

Finally in the last section we analyze the Auslander-Reiten quivers of Nakayama algebras given as path algebras  $kA_n/I$  for some ideal  $I \trianglelefteq kA_n$ . The first class are such Nakayama algebras with global dimension  $n - 1$ . In this case the Auslander-Reiten quiver consists of only one component  $\mathbb{Z}[A_n]$ . So by previous results the algebra  $kA_n/I$  is derived equivalent to  $A_n$ .

The second class are the algebras where  $I$  is generated by the path of length  $n - 1$ . In the second case the Auslander-Reiten quiver consists of only one component  $\mathbb{Z}[D_n]$ . Therefore the algebra  $kA_n/I$  is derived equivalent to  $kD_n$ .

The notation is the same as in [S]. Also some definitions and results from [S] will be used. For the convenience of the reader the notation and definitions are stated in this paper.

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## 2. PRELIMINARIES AND NOTATION

Let  $A$  denote a finite-dimensional indecomposable algebra over a field  $k$  and  $A\text{-mod}$  the category of finite-dimensional left  $A$ -modules. We denote by  $\mathcal{P}$  the full subcategory in  $A\text{-mod}$  of projective modules and  $\mathcal{I}$  the full subcategory category of injective  $A$ -modules. Let  $C \in \{A\text{-mod}, \mathcal{P}, \mathcal{I}\}$ .

Then  $Comp^{*,?}(C)$  is the category of complexes that are bounded above if  $* = -$ , bounded below if  $* = +$  and bounded if  $* = b$ . The homology is bounded if  $? = b$ . We denote by  $D^b(A)$  the bounded derived category and by  $K^{*,?}(C)$  the homotopy category.

The homotopy category and the derived categories are triangulated categories by [Wei, 10.2.4, 10.4.3] where the shift functor  $[1]$  is the automorphism. The distinguished triangles are given up to isomorphism of triangles by

$$X \xrightarrow{f} Y \xrightarrow{0 \oplus \text{id}_Y} \text{cone}(f) \xrightarrow{\text{id}_{X[1]} \oplus 0} X[1].$$

for any morphism  $f$ .

It is difficult to calculate the morphisms in the derived category of  $A$ -modules. The following theorem provides an easier way to represent them.

**Theorem 2.1.** [Wei, 10.4.8] *We have the following equivalences of triangulated categories*

$$K^{-,b}(\mathcal{P}) \cong K^{+,b}(\mathcal{I}) \cong D^b(A).$$

We identify a  $A$ -module  $X$  with the complex that has entry  $X$  in degree 0 and entry 0 in all other degrees. By abuse of notation we call this complex  $X$ . A complex with non-zero entry in only one degree is also called a stalk complex. Note that  $A\text{-mod}$  is equivalent to a full subcategory of  $D^b(A)$  using this embedding.

Let  $N$  be a left  $A$ -module and  $\cdots \xrightarrow{d_P^2} P_1 \xrightarrow{d_P^1} P_0 \rightarrow N$  its minimal projective resolution. Let  $N \rightarrow I_0 \xrightarrow{d_I^0} I_1 \xrightarrow{d_I^1} \cdots$  be its minimal injective resolution. Then we denote throughout this paper by  $pN$  the complex with  $(pN)^i = P_{-i}$  and  $d^i := d_P^{-i}$  for  $i \leq 0$  and  $(pN)^i = 0$  for  $i > 0$ . Similarly we define  $iN$  to be the complex with  $(iN)^n = I_n$  and  $d^n := d_I^n$  for  $n \geq 0$  and  $(iN)^n = 0$  for  $n < 0$ .

**Remark 2.2.** *Note that all indecomposable contractible complexes in  $\text{Comp}^{-,b}(\mathcal{P})$  have up to shift the form*

$$\cdots \rightarrow 0 \rightarrow P \xrightarrow{\text{id}} P \rightarrow 0 \rightarrow \cdots$$

*for an indecomposable projective module  $P$  of  $A$ . We denote such a complex where  $P$  occurs in degree 0 and 1 by  $\bar{P}$ .*

Contractible complexes are projective objects in  $\text{Comp}^{-,b}(\mathcal{P})$  (see for example [S, 2.9]).

Finally we define for a complex  $X$ , the complex  $\sigma^{\leq n}(X)$  to be the complex with  $\sigma^{\leq n}(X)^i = X^i$  for  $i \leq n$  and  $d_{\sigma^{\leq n}(X)}^i = d_X^i$  for  $i < n$  and  $\sigma^{\leq n}(X)^i = 0$  for  $i > n$ . We define  $\sigma^{\geq n}(X)$  analogously.

Next we introduce Auslander-Reiten theory for  $D^b(A)$ . We state the existence conditions for Auslander-Reiten triangles in the bounded derived category of a finite-dimensional algebra and prove some properties that will be needed in the other sections.

For an introduction to triangulated categories we refer to [H2, 1.1].

**Definition 2.3.** [H1, 4.1] *[Auslander-Reiten triangles] A distinguished triangle  $X \rightarrow_u Y \rightarrow_v Z \rightarrow_w X[1]$  is called an Auslander-Reiten triangle if the following conditions are satisfied:*

- (1) *The objects  $X, Z$  are indecomposable*
- (2) *The map  $w$  is non-zero*
- (3) *If  $f : W \rightarrow Z$  is not a retraction, then there exists  $f' : W \rightarrow Y$  such that  $v \circ f' = f$ .*

By [H1, 4.2] we have the following equivalences. The condition (3) is equivalent to

- (3'') *If  $f : W \rightarrow Z$  is not a retraction, then  $w \circ f = 0$ .*

We refer to  $w$  as the connecting homomorphism of an Auslander-Reiten triangle. We say that the Auslander-Reiten triangle  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$  starts in  $X$ , has middle term  $Y$  and ends in  $Z$ . Note also that an Auslander-Reiten triangle is uniquely determined up to isomorphisms of triangles by the isomorphism class of the element it ends or starts with. The Auslander-Reiten translation  $\tau$  is the functor from the set of isomorphism classes of indecomposable objects that appear at the end of an Auslander-Reiten triangle to the set of isomorphism classes of indecomposable objects at the start of an Auslander-Reiten triangle. The functor  $\tau$  sends the isomorphism class of  $Z$  to the isomorphism class of  $X$ .

The Auslander-Reiten translation of  $A\text{-mod}$  will be denoted by  $\tau_A$  in this chapter to avoid confusion.

Analogously to the classical Auslander-Reiten theory we can define irreducible maps, minimal maps, left almost split maps and right almost split maps as in [ASS, 1.1,1.4]. Irreducible maps here have the same properties as in the case of Artin algebras. (see [ASS, 1.8,1.10])

**Lemma 2.4.** *Let  $N, M \in D^b(A)$  and let  $f : N \rightarrow M$  be an irreducible map in  $D^b(A)$ .*

(1) *Let  $N \rightarrow_g Q \rightarrow E \rightarrow TN$  be the Auslander-Reiten triangle, then there is a retraction  $s : Q \rightarrow M$  such that  $f = s \circ g$ .*

(2) *Let  $L \rightarrow B \rightarrow_h M \rightarrow TL$  be an Auslander-Reiten triangle, then there is a section  $r : N \rightarrow B$  such that  $f = h \circ r$ .*

We define the Auslander-Reiten quiver to be the labelled graph  $\Gamma(D^b(A))$  with vertices the isomorphism classes of indecomposable objects. The label of an arrow  $X \xrightarrow{(d_{XY}, d'_{XY})} Y$  is defined as follows. If there is an Auslander-Reiten triangle  $X \rightarrow L \rightarrow \tau^{-1}(X) \rightarrow X[1]$  then  $d_{XY}$  is the multiplicity of  $Y$  as a direct summand of  $L$ . Analogously if there is an Auslander-Reiten triangle  $\tau(Y) \rightarrow M \rightarrow Y \rightarrow \tau(Y)[1]$  ending in  $Y$ , then  $d'_{XY}$  is the multiplicity of  $X$  as a direct summand of  $M$ . We call a connected component of  $\Gamma(D^b(A))$  an Auslander-Reiten component. For well-definedness we refer to [H1, ].

**Lemma 2.5.** [H1, 4.3, 4.5] *Let  $X \rightarrow_u M \rightarrow_v Z \rightarrow X[1]$  be an Auslander-Reiten triangle and  $M \cong M_1 \oplus M_2$ , where  $M_1$  is indecomposable. Let  $i : M_1 \rightarrow M$  be an inclusion and  $p : M \rightarrow M_1$  a projection. Then  $v \circ i : M_1 \rightarrow Z$  and  $p \circ u : X \rightarrow M$  are irreducible maps. Furthermore  $u$  is minimal left almost split and  $v$  is minimal right almost split.*

Let  $\nu_A$  denote the Nakayama functor of  $A$  and let

$$\nu_A^{-1} := \text{Hom}_A(\text{Hom}_k(-, k), A).$$

We denote by  $\nu$  the left derived functor of  $\nu_A$  on  $D^b(A)$  and by  $\nu^{-1}$  the right derived functor of  $\nu_A^{-1}$ . Then  $\nu$  maps a complex  $X \in \text{Comp}^b(\mathcal{P})$  to  $\nu(X) \in \text{Comp}^b(\mathcal{I})$ , where  $\nu(X)^i := \nu_A(X^i)$  and  $d_{\nu(X)}^i := \nu_A(d_X^i)$  for all  $i \in \mathbb{Z}$ .

For  $D^b(A)$  the existence of Auslander-Reiten triangles have been determined.

**Theorem 2.6.** [H2, 1.4] (1) Let  $Z \in K^{-,b}(\mathcal{P})$  be indecomposable. Then there exists an Auslander-Reiten triangle ending in  $Z$  if and only if  $Z \in K^b(\mathcal{P})$ . The triangle is of the form  $\nu(Z)[-1] \rightarrow Y \rightarrow Z \rightarrow \nu(Z)$  for some  $Y \in K^{-,b}(\mathcal{P})$ .

(2) Let  $X \in K^{+,b}(\mathcal{I})$  be indecomposable, then there exists an Auslander-Reiten triangle starting in  $X$  if and only if  $X \in K^b(\mathcal{I})$ . The triangle is of the form  $X \rightarrow Y \rightarrow \nu^{-1}(X)[1] \rightarrow X[1]$  for some  $Y \in K^{-,b}(\mathcal{P})$ .

From this result we deduce that the translation  $\tau$  is given by  $\nu[-1]$  and  $\tau$  is natural equivalence from  $K^b(\mathcal{P})$  to  $K^b(\mathcal{I})$ .

Let  $N, M \in D^b(A)$  be two indecomposable elements and let  $f : N \rightarrow M$  be an irreducible map. Then there is an arrow from  $N$  to  $M$  in the Auslander-Reiten quiver representing  $f$  if and only if  $N \in K^b(\mathcal{I})$  or  $M \in K^b(\mathcal{P})$ .

### 3. AUSLANDER-REITEN TRIANGLES

In this section we establish some properties of Auslander-Reiten triangles and definitions needed for the rest of this paper.

We call an Auslander-Reiten component  $\Lambda$  stable, if  $\tau$  is an automorphism on  $\Lambda$ . By 2.6 this is equivalent to the fact that all vertices in  $\Lambda$  are in  $K^b(\mathcal{I})$  and  $K^b(\mathcal{P})$ .

By [XZ, 2.2.1] and [Ri, p.206] we have

**Corollary 3.1.** Let  $\Lambda$  be a stable Auslander-Reiten component of  $D^b(A)$ . Then  $\Lambda \cong \mathbb{Z}[T]/I$  where  $T$  is a tree and  $I$  is an admissible subgroup of  $\text{aut}(\mathbb{Z}[T])$ .

The following lemma determines the relation between irreducible maps, retractions and sections in  $K^{-,b}(\mathcal{P})$  and  $\text{Comp}^{-,b}(\mathcal{P})$ . Note that by duality the same is true if we replace  $K^{-,b}(\mathcal{P})$  by  $K^{+,b}(\mathcal{I})$  and  $\text{Comp}^{-,b}(\mathcal{P})$  by  $\text{Comp}^{+,b}(\mathcal{I})$ .

**Lemma 3.2.** [S, 3.1] Let  $B, C \in \text{Comp}^{-,b}(\mathcal{P})$  be complexes that are not contractible. Let  $f : B \rightarrow C$  be a map of complexes.

(1) Let  $C, B$  be indecomposable. The map  $f$  is irreducible in  $\text{Comp}^{-,b}(\mathcal{P})$  if and only if  $f$  is irreducible in  $K^{-,b}(\mathcal{P})$ .

(2) Let  $C$  be indecomposable. The map  $f$  is a retraction in  $\text{Comp}^{-,b}(\mathcal{P})$  if and only if  $f$  is a retraction in  $K(\mathcal{P})$ .

(3) Let  $B$  be indecomposable. The map  $f$  is a section in  $\text{Comp}^{-,b}(\mathcal{P})$  if and only if  $f$  is a section in  $K^{-,b}(\mathcal{P})$ .

We can therefore choose for an irreducible map in  $K^{-,b}(\mathcal{P})$  an irreducible map in  $\text{Comp}^{-,b}(\mathcal{P})$  that represents this map. For the rest of this chapter all irreducible maps in  $K^{-,b}(\mathcal{P})$  or  $K^{+,b}(\mathcal{I})$  will be represented by irreducible maps in  $\text{Comp}^{-,b}(\mathcal{P})$  or  $\text{Comp}^{+,b}(\mathcal{I})$  respectively.

The following straightforward result will be used often to show that a complex is indecomposable.

**Lemma 3.3.** *Let  $X \in \text{Comp}(A)$  be such that  $X^i$  is indecomposable for all  $i \in \mathbb{Z}$  and  $d_X^l = 0$  if and only if  $X^j = 0$  for all  $j > l$  or  $X^j = 0$  for all  $j \leq l$ . Then  $X$  is an indecomposable complex.*

The next result follows from the previous lemma by applying 3.2.

**Corollary 3.4.** *Let  $X \in \text{Comp}^{-,b}(\mathcal{P})$  ( respectively  $\text{Comp}^{+,b}(\mathcal{I})$ ) such that  $X^i$  is indecomposable for all  $i \in \mathbb{Z}$  and  $d_X^l = 0$  if and only if  $X^j = 0$  for all  $j > l$  or  $X^j = 0$  for all  $j \leq l$ . Then  $X$  is an indecomposable object in  $K^{-,b}(\mathcal{P})$  (respectively  $K^{+,b}(\mathcal{I})$ ).*

We can determine the homology of the middle term of an Auslander-Reiten triangle ending in the stalk complex of a projective indecomposable module.

**Lemma 3.5.** *Let  $P$  be a projective indecomposable module and let  $M$  be the middle term of the Auslander-Reiten triangle ending in  $P$ . Then  $H^1(M) = I/\text{soc } I$  where  $I := \nu(P)$ ,  $H^0(M) = \text{rad } P$  and  $H^i(M) = 0$  for all  $i \neq 0, 1$ .*

*Proof.* Let  $I := \nu_A(P)$  and let  $\cdots \rightarrow P_2 \rightarrow_g P_1 \rightarrow_f P_0$  be a minimal projective resolution of  $I$ . Then  $I$  is isomorphic to  $pI$  in  $D^b(A)$ . Let  $w : P \rightarrow pI$  be a homomorphism in  $\text{Comp}^{-,b}(\mathcal{P})$  representing the Auslander-Reiten triangle ending in  $P$ . By 2.6 the Auslander-Reiten triangle can be written as  $pI[-1] \rightarrow \text{cone}(w)[-1] \rightarrow P \rightarrow_w pI$ . Let  $M = \text{cone}(w)[-1]$ , then  $M$  is given by

$$\cdots \rightarrow P_2 \xrightarrow{g} P \oplus P_1 \xrightarrow{(f, w^0)} P_0 \rightarrow 0 \rightarrow \cdots$$

where  $P \oplus P_1$  appears in degree zero. We have  $H^1(M) \cong P_0/\text{Im}(f, w^0)$  and  $\text{Im}(f, w^0) = \text{Im } f + \text{Im } w^0$ . We show next that  $\text{Im}(f, w^0)/\text{Im } f$  is simple, hence isomorphic to  $\text{soc } I$ . Let  $h : P' \rightarrow P$  be a projective cover of  $\text{rad } P$ . We identify  $h$  with the corresponding map of complexes  $P' \rightarrow P$ . As  $h$  is not a retraction we have  $w \circ h = 0$  in  $K^{-,b}(\mathcal{P})$  by 2.3 (3'') and  $w \circ h$  is therefore homotopic to zero. Then there is a map  $s : P' \rightarrow P_1$  such that  $w^0 \circ h = f \circ s$ . We visualize this in the following diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & P' & \longrightarrow & 0 \longrightarrow \cdots \\
& & \downarrow & & \downarrow h & & \downarrow \\
\cdots & \longrightarrow & 0 & \xrightarrow{s} & P & \longrightarrow & 0 \longrightarrow \cdots \\
& & \downarrow & & \downarrow w^0 & & \downarrow \\
\cdots & \longrightarrow & P_1 & \xrightarrow{f} & P_0 & \longrightarrow & 0 \longrightarrow \cdots
\end{array}$$

As the diagram commutes, we have  $w^0(\text{rad } P) \subset \text{Im } f$  and  $w^0(P) \not\subset \text{Im } f$ . As  $P/\text{rad } P$  is simple,  $\text{Im}(f, w^0)/\text{Im } f$  is also simple and we have  $H^1(M) = I/\text{soc } I$ . Furthermore  $\ker(f, w^0) = \text{rad } P \oplus \text{Im } g$ . Therefore  $H^0(M) = \text{rad } P$ . Clearly  $H^i(M) = 0$  for all  $i \neq 0, 1$ .  $\square$

We generalize Wheeler's construction [W, 2.4] in the next theorem.

**Theorem 3.6.** *Let  $P$  be a projective indecomposable module that is not simple and suppose  $\nu_A^i(P)$  is injective and projective for all  $i \in \mathbb{Z}$ . Then the Auslander-Reiten component  $\Lambda$  of  $D^b(A)$  containing  $P$  is isomorphic to  $\mathbb{Z}[A_\infty]$ .*

*Proof.* Let  $w$  be the connecting map in the Auslander-Reiten triangle ending in  $P$ . We can choose  $w$  such that  $w^0$  induces a projection of  $P/\text{rad } P$  onto  $\text{soc } \nu_A(P)$  by 3.5. Set  $f^i := \nu_A^i(w^0)$  for all  $i \in \mathbb{Z}$ . The elements

$$P_i := \cdots \rightarrow 0 \rightarrow P \xrightarrow{f^0} \nu_A(P) \rightarrow \cdots \rightarrow \nu_A^{i-1}(P) \xrightarrow{f^{i-1}} \nu_A^i(P) \rightarrow 0 \rightarrow \cdots$$

with entry  $P$  in degree 0 are complexes by the choice of  $w^0$  and they are indecomposable for all  $i \in \mathbb{Z}$  by 3.4. We set  $P_{-1} = 0$ . The Auslander-Reiten triangles in  $\Lambda$  are given by

$$\nu_A(P_{i-1})[-1] \rightarrow P_i \oplus \nu_A(P_{i-2})[-1] \rightarrow P_{i-1}.$$

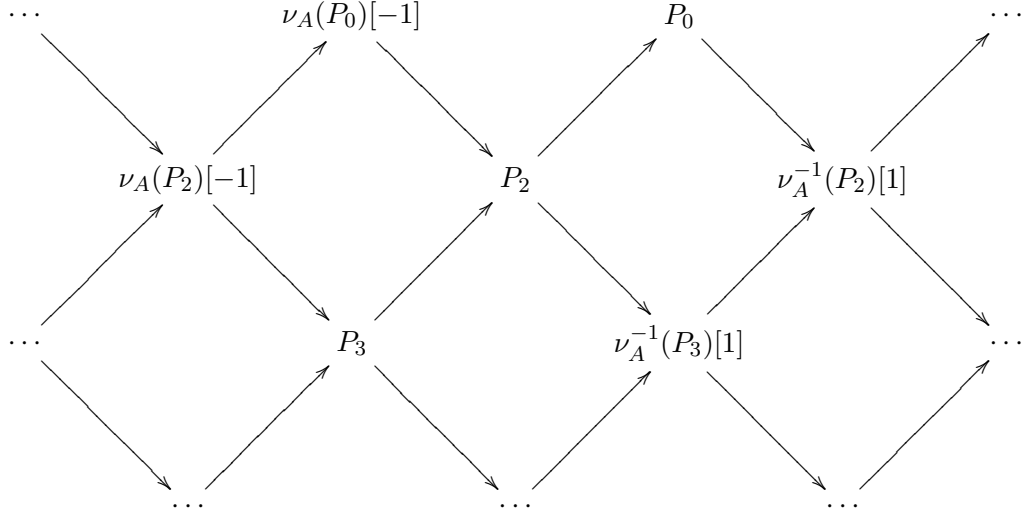
This can be seen as follows: the middle term of the Auslander-Reiten sequence ending in  $P_{i-1}$  can be taken as the upper row of the next diagram and the bottom row is the direct summand  $P_i$ .

$$\begin{array}{ccccccc}
0 & \longrightarrow & P & \xrightarrow{(f^0, 0)} & \nu_A(P) \oplus \nu_A(P) & \xrightarrow{\begin{pmatrix} f^1, 0 \\ 0, -f^1 \end{pmatrix}} & \cdots \longrightarrow \nu_A^{i-1}(P) \oplus \nu_A^{i-1}(P) \xrightarrow{\begin{pmatrix} -f^{i-1}, 0 \\ 0, -f^{i-1} \end{pmatrix}} \nu_A^i(P) \longrightarrow 0 \\
& & \text{id} \downarrow \uparrow \text{id} & & (\text{id}, \pm \text{id}) \downarrow \uparrow \text{id} & & (\text{id}, \text{id}) \downarrow \uparrow \text{id} & & -\text{id} \downarrow \uparrow -\text{id} \\
0 & \longrightarrow & P & \xrightarrow{f^0} & \nu_A(P) & \xrightarrow{f^2} & \cdots \longrightarrow \nu_A^{i-1}(P) \xrightarrow{f^{i-1}} \nu_A^i(P) \longrightarrow 0
\end{array}$$

We have to alternate the signs of the second identity map from column two to  $i - 1$  so that we have a positive sign in the  $i$ -th column for all



$i \in \mathbb{Z}$ . This sequence has as direct summand  $P_i$  and  $\nu_A(P_{i-2})[-1]$ . Therefore  $\Lambda \cong \mathbb{Z}[A_\infty]$ , and  $\Lambda$  looks as follows:



As  $\nu_A^i(P)[-i] \not\cong P$  the component cannot be a tube.  $\square$

For an self-injective algebra we have  $\mathcal{P} = \mathcal{I}$  and we can therefore apply the previous theorem to construct Auslander-Reiten components. We can also use this theorem to compute Auslander-Reiten components of non-self-injective algebras, as shown in the Example 3.10.

Next we define certain length functions of complexes similar to the functions defined in [W].

**Definition 3.7.** For  $Y \in \text{Comp}^b(\mathcal{P})$  we denote by  $l(Y)$  the number of projective indecomposable summands in  $\oplus_{i \in \mathbb{Z}} Y^i$ . For  $X \in K^b(\mathcal{P})$  we denote by

$$l_p(X) := \min\{l(Y) \mid Y \in \text{Comp}^b(\mathcal{P}) \text{ and } Y \cong X \text{ in } K^b(\mathcal{P}).\}$$

For well-definedness see [S, 4.5].

**Remark 3.8.** Let  $\Lambda$  be a stable component. Then by 2.6 every Auslander-Reiten triangle is isomorphic to

$$\tau(P) \rightarrow \text{cone}(w)[-1] \rightarrow P \xrightarrow{w} \tau(P)[1]$$

where

$$0 \rightarrow \tau(P) \rightarrow \text{cone}(w)[-1] \rightarrow P \rightarrow 0$$

is a short exact sequence in  $\text{Comp}^b(\mathcal{P})$  and  $w : P \rightarrow \tau(P)[1]$  is a representative in  $\text{Comp}^b(\mathcal{P})$  of the connecting morphism. Then  $\text{cone}(w)[-1] \in \text{Comp}^b(\mathcal{P})$ . Therefore  $l_p$  is well-defined on a stable component and satisfies

$l_p(\text{cone}(w)[-1]) \leq l_p(\tau(P)) + l_p(P)$  with equality if and only if  $\text{cone}(w)[-1]$  does not have a contractible summand in  $\text{Comp}^b(\mathcal{P})$ .

We can generalize [W, 3.2] where the same result is proven in the case that  $A$  is a self-injective algebra. The next result also generalizes 3.6 which has only been proven for stalk complexes of projective modules.

**Theorem 3.9.** *Let  $X \in \text{Comp}^b(\mathcal{P})$  be an indecomposable not contractible complex such that  $\nu_A^i(X^j)$  is projective and injective for all  $i, j \in \mathbb{Z}$ . Then  $X$  is in an Auslander-Reiten component of  $D^b(A)$  isomorphic to  $\mathbb{Z}[A_\infty]$  or  $A_1$ .*

*Proof.* Let  $C$  be the Auslander-Reiten component of  $\Gamma(D^b(A))$  containing  $X$  with tree class  $T$ . The function  $l_p$  is constant on all  $\tau$ -orbits and therefore  $l_p$  is a subadditive function on  $T$ . Suppose  $l_p$  is bounded. Then by [S, 4.15] the algebra  $A$  is derived equivalent to  $D^b(kQ)$  for a finite Dynkin diagram  $Q \neq A_1$  or  $A$  is simple. As in the first case there is no complex that satisfies the assumption, we have that  $A$  is simple. So  $C \cong A_1$ . Otherwise  $l_p$  is unbounded and  $C$  has therefore tree class  $A_\infty$ . Assume without loss of generality that  $X^i \neq 0$  if and only if  $0 \leq i \leq n$ . We have  $H^0(X) \neq 0$  as  $X$  does not have contractible summands. Then  $\tau(X)$  viewed as complex in  $\text{Comp}^b(\mathcal{P})$  satisfies  $\tau(X)^i \neq 0$  if and only if  $1 \leq i \leq n+1$ . Therefore  $X$  is not periodic and  $C \cong \mathbb{Z}[A_\infty]$ .  $\square$

We compute some examples of Auslander-Reiten quivers using the previous results.

**Example 3.10.** *Let  $k$  be a field and let  $G$  be the quiver*

$$1 \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{array} 2$$

*Let  $A := kG/R$  where  $kG$  is the path algebra of  $G$ , and where  $R$  is the ideal generated by  $\{\alpha\beta\}$ . The algebra  $A$  has global dimension 2 and has finite representation type. We denote by  $S_1$  and  $S_2$  the simple modules corresponding to the vertices 1 and 2. Let  $P_i$  be the projective covers of  $S_i$ . We have  $S_1 = P_1/P_2$  and  $P_2/S_1 = S_2$ . We fix a non-zero map  $f : P_1 \rightarrow P_1$  that maps the generator to the socle. Then  $f$  is not an isomorphism. The derived category  $D^b(A)$  has infinitely many indecomposable elements  $I_n$  given by  $I_n^m = P_1$  for all  $0 \leq m \leq n-1$  and  $d^l = f$  for  $0 \leq l \leq n-2$  for all  $0 \leq n \leq n-2$ . By [BGS, Theorem A] we have that  $D^b(A)$  is discrete.*

As  $I_1 = P_1$  we have by 3.6 that all  $I_n$  are the elements of a component  $\mathbb{Z}[A_\infty]$ .

As we have  $\tau(S_2) = S_2[1]$  the element  $S_2$  belongs to a component  $\mathbb{Z}[A_\infty]$  by [S, 4.13] and [S, 4.14].

Finally we note that the  $\tau$ -orbit of  $P_2$  is given by

$$\tau^n(P_2)^i = \begin{cases} P_1, & \text{for } -n \leq i \leq n \\ P_2, & \text{for } i = -n - 1 \\ 0 & \text{else.} \end{cases}$$

Therefore the value of  $l_p$  is strictly increasing on  $\tau$ -orbits of the component containing  $P_2$ . The predecessors of  $P_2$  are  $S_1$  and  $S_1[-1]$ . Therefore the component is  $\mathbb{Z}[A_\infty]$ . There are no components of Euclidean tree class, as these would have to contain at least one simple  $A$ -module by 4.6.

The previous example is a discrete derived categories. These categories have been defined and classified in [V, Theorem] and their Auslander-Reiten quivers have been determined in [BGS, Theorem A]. The authors do not calculate the Auslander-Reiten quiver directly but use the fact that the Happel functor (see [H2, 2.5]) induces an equivalence of triangulated categories between  $D^b(A)$  and the stable module category of the repetitive algebra  $\widehat{A}$  for all algebras  $A$  of finite-global dimension by [H2, 2.3 Theorem].

We can also determine the predecessors for some projective indecomposable modules  $P$  using the previous results.

**Corollary 3.11.** (1) If  $\nu_A(P)$  is projective, then

$$\cdots \rightarrow 0 \rightarrow P \rightarrow \nu_A(P) \rightarrow 0 \rightarrow \cdots$$

with  $P$  appearing in degree 0, is the only predecessor of  $P$ .

(2) If  $P$  is injective, then

$$\cdots \rightarrow 0 \rightarrow P \rightarrow \nu_A(P) \rightarrow 0 \rightarrow \cdots$$

with  $P$  appearing in degree 0 is the only predecessor of  $P$ .

*Proof.* Let  $w$  be the connecting homomorphism in the Auslander-Reiten triangle ending in  $P$ . Suppose that  $\nu_A(P)$  is projective. Then  $\text{cone}(w)$  can be taken as  $\cdots \rightarrow 0 \rightarrow P \xrightarrow{w^0} \nu_A(P) \rightarrow 0 \rightarrow \cdots$ , where  $w^0$  induces a projection of  $P/\text{rad } P$  onto  $\text{soc } \nu_A(P)$  by 3.5. The complex  $\text{cone}(w)$  is indecomposable in  $K^b(\mathcal{P})$  by 3.4. This proves part (1) and part (2) follows similarly.  $\square$

## 4. IRREDUCIBLE MAPS ENDING IN CONTRACTIBLE COMPLEXES

Next we analyze under which conditions a contractible complex can appear as direct summand of  $\text{cone}(w)[-1] \in \text{Comp}^{-,b}(\mathcal{P})$  for a map  $w$  in  $\text{Comp}^{-,b}(\mathcal{P})$  that induces an Auslander-Reiten triangle  $\nu(Z)[-1] \rightarrow Y \rightarrow Z \xrightarrow{w} \nu(Z)$  in  $D^b(A)$ .

We first introduce a new definition.

**Definition 4.1.** Let  $P_1, P_2 \in \mathcal{P}$  and  $f : P_1 \rightarrow P_2$  be a map. Then  $f$  is  $p$ -irreducible if  $f$  is not a section and not a retraction and if for any  $P \in \mathcal{P}$  and maps  $f_1 : P_1 \rightarrow P$  and  $f_2 : P \rightarrow P_2$  such that  $f = f_2 \circ f_1$  we have that  $f_1$  is a section or  $f_2$  is a retraction.

Throughout this section let  $P$  be an indecomposable projective module and  $\bar{P}$  the contractible complex

$$\cdots \rightarrow 0 \rightarrow P \xrightarrow{\text{id}} P \rightarrow 0 \rightarrow \cdots$$

with  $\bar{P}^0 = \bar{P}^1 = P$ .

**Lemma 4.2.** Let  $f : Q \rightarrow \bar{P}$  be an irreducible map in  $\text{Comp}^{-,b}(\mathcal{P})$ , where  $Q \in \text{Comp}^{-,b}(\mathcal{P})$  is indecomposable and not contractible. Then there exists an indecomposable projective module  $P_0$  and a map  $d : P_0 \rightarrow P$  that is  $p$ -irreducible, such that  $Q \cong p(\text{Coker}(d))$ .

*Proof.* Let  $f$  be an irreducible map given by the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & Q^{-1} & \longrightarrow & Q^0 & \xrightarrow{d} & Q^1 & \longrightarrow & Q^2 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow f^0 & & \downarrow f^1 & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & P & \xrightarrow{\text{id}} & P & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

We first show that  $Q^i = 0$  for  $i \geq 2$ , and that  $Q^1 \cong P$ . We can factorize  $f$  through  $\sigma^{\leq 1}Q$  as

$$\begin{array}{ccccccc} \cdots & \longrightarrow & Q^{-1} & \longrightarrow & Q^0 & \xrightarrow{d} & Q^1 & \longrightarrow & Q^2 & \longrightarrow & \cdots \\ & & \downarrow \text{id} & & \downarrow \text{id} & & \downarrow \text{id} & & \downarrow & & \\ \cdots & \longrightarrow & Q^{-1} & \longrightarrow & Q^0 & \xrightarrow{d} & Q^1 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow f^0 & & \downarrow f^1 & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & P & \xrightarrow{\text{id}} & P & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

The map given by the last two rows is not a retraction, as  $f$  is not a retraction. Therefore the map between the first two rows is a section and  $Q^i = 0$  for all  $i \geq 2$ .

We can factorize  $f$  as

$$\begin{array}{ccccccccc}
 \cdots & \longrightarrow & Q^{-1} & \longrightarrow & Q^0 & \xrightarrow{d} & Q^1 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow d & & \downarrow \text{id} & & \downarrow & & \\
 \cdots & \longrightarrow & 0 & \longrightarrow & Q^1 & \xrightarrow{\text{id}} & Q^1 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow f^1 & & \downarrow f^1 & & \downarrow & & \\
 \cdots & \longrightarrow & 0 & \longrightarrow & P & \xrightarrow{\text{id}} & P & \longrightarrow & 0 & \longrightarrow & \cdots
 \end{array}$$

If the map between the first two rows is a section, then  $Q$  is isomorphic to the complex in the middle row, and hence is contractible, which is a contradiction. Therefore the map between the last two rows is a retraction. We can write  $Q^1 \cong P \oplus P'$  for a projective module  $P'$  and  $f^1$  is a retraction. But then we can factorize  $f$  as follows

$$\begin{array}{ccccccccc}
 \cdots & \longrightarrow & Q^{-1} & \longrightarrow & Q^0 & \xrightarrow{d} & P \oplus P' & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow \text{id} & & \downarrow \text{id} & & \downarrow f^1 & & \downarrow & & \\
 \cdots & \longrightarrow & Q^{-1} & \longrightarrow & Q^0 & \xrightarrow{f^1 d} & P & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow f^0 & & \downarrow \text{id} & & \downarrow & & \\
 \cdots & \longrightarrow & 0 & \longrightarrow & P & \xrightarrow{\text{id}} & P & \longrightarrow & 0 & \longrightarrow & \cdots
 \end{array}$$

If the map in the last two rows is a retraction, then  $f$  is a retraction. Therefore the map between the two upper rows has to be a section and  $P' = 0$ ,  $f^0 = d$  and  $f^1 = \text{id}$ .

Let  $\cdots \rightarrow L_{-2} \rightarrow L_{-1} \rightarrow \ker d$  be a minimal projective resolution of  $\ker d$ . Then  $f$  factorizes through  $\text{Coker}(d)$  as follows

$$\begin{array}{ccccccccc}
 \cdots & \longrightarrow & Q^{-1} & \xrightarrow{h} & Q^0 & \xrightarrow{d} & P & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow \text{id} & & \downarrow \text{id} & & \downarrow & & \\
 \cdots & \longrightarrow & L_{-1} & \xrightarrow{g} & Q^0 & \xrightarrow{d} & P & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow d & & \downarrow \text{id} & & \downarrow & & \\
 \cdots & \longrightarrow & 0 & \longrightarrow & P & \xrightarrow{\text{id}} & P & \longrightarrow & 0 & \longrightarrow & \cdots
 \end{array}$$

Clearly the bottom diagram is not a retraction, because else  $f$  would be a retraction. Therefore the upper diagram is a section and the  $Q^{-r}$  are

isomorphic to direct summands of  $L_{-r}$  for all  $r > 0$ . Then there exists a projection  $s : L_{-1} \rightarrow Q^{-1}$  such that  $hs = g$ . By the minimality of the projective resolution, we have  $Q^{-1} \cong L_{-1}$ . By induction  $Q$  is isomorphic to the complex consisting of a minimal projective resolution of  $\text{Coker } d$  and 0 elsewhere. Suppose  $d$  is not  $p$ -irreducible. Then we can factorize  $d = s \circ t$  where  $t : Q^0 \rightarrow \tilde{P}$  is not a section and  $s : \tilde{P} \rightarrow P$  is not a retraction for some projective module  $\tilde{P}$ . But then we can factorize  $f$  as follows

$$\begin{array}{ccccccccc}
 \cdots & \longrightarrow & Q^{-1} & \xrightarrow{h} & Q^0 & \xrightarrow{d} & P & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \text{id} \downarrow & & t \downarrow & & \text{id} \downarrow & & \downarrow & & \\
 \cdots & \longrightarrow & Q^{-1} & \xrightarrow{th} & \tilde{P} & \xrightarrow{s} & P & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & s \downarrow & & \text{id} \downarrow & & \downarrow & & \\
 \cdots & \longrightarrow & 0 & \longrightarrow & P & \xrightarrow{\text{id}} & P & \longrightarrow & 0 & \longrightarrow & \cdots
 \end{array}$$

This is a contradiction to the irreducibility of  $f$  as the map between the first two rows is not a section and the map between the two bottom rows is not a retraction. Therefore  $d$  is  $p$ -irreducible.  $\square$

Next we determine  $p$ -irreducible maps between indecomposable projective modules.

**Lemma 4.3.** *Let  $P_1$  and  $P_2$  be indecomposable projective modules and  $d : P_1 \rightarrow P_2$  some map. Then  $d$  is  $p$ -irreducible if and only if  $P_1$  is isomorphic to a direct summand of the projective cover of  $\text{rad } P_2$  and  $d$  is induced by this projection.*

*Proof.*  $\Leftarrow$

Let  $d : P_1 \rightarrow P_2$  be as in the statement. Suppose that  $d$  factors as  $d = g \circ f$  where  $f : P_1 \rightarrow \tilde{P}$ ,  $g : \tilde{P} \rightarrow P_2$  and  $g$  is not a retraction. Then  $\text{Im } d \subset \text{Im } g \subset \text{rad } P_2$ . Let  $\pi : P_1 \oplus P'_1 \rightarrow \text{rad } P_2$  be the projective cover. Since  $\tilde{P}$  is projective, there is a map  $e : \tilde{P} \rightarrow P_1 \oplus P'_1$  such that  $\pi e = g$ . Then we have a commutative diagram

$$\begin{array}{ccccc}
 P_1 & \xrightarrow{f} & \tilde{P} & \xrightarrow{e} & P_1 \oplus P'_1 \\
 d \downarrow & & g \downarrow & & \downarrow \\
 \text{Im } d & \longrightarrow & \text{Im } g & \longrightarrow & \text{rad } P_2
 \end{array}$$

Let  $\text{top } P_1 \cong S_1$  and let  $q : \text{rad } P_2 \rightarrow S_1$  be a projection with  $q(\text{Im } d) \neq 0$ . Then  $q(\text{Im}(ef)) \neq 0$ . Let  $p : P_1 \oplus P'_1 \rightarrow P_1$  denote the natural projection.

Then  $pef \notin \text{rad } P_1$ . Therefore  $pef$  is an isomorphism. So  $f$  is a section and  $d$  is  $p$ -irreducible.

$\implies$

Conversely, let  $s : P_1 \rightarrow P_2$  be a  $p$ -irreducible map. Then  $s$  is not surjective and therefore  $\text{Im } s \subset \text{rad } P_2$ . Let  $P_0$  be a projective module and  $i : P_0 \rightarrow \text{rad } P_2$  be a projection. Then there is a map  $h : P_1 \rightarrow P_0$  such that  $s = ih$ . As  $s$  is  $p$ -irreducible,  $h$  is a section and  $P_1$  is a direct summand of  $P_0$ .  $\square$

Next we determine which  $p$ -irreducible maps induce an irreducible map in  $\text{Comp}^{-,b}(\mathcal{P})$ .

**Theorem 4.4.** *Let  $P_0$  be a projective module and let  $d : P_0 \rightarrow P$  be a  $p$ -irreducible map. Then there is an irreducible map in  $\text{Comp}^{-,b}(\mathcal{P})$  from the complex  $p(\text{Coker}(d))$  to the contractible complex  $\bar{P}$  if and only if  $P_0$  is the projective cover of  $\text{rad } P$  and  $d$  is the projection onto  $\text{rad } P$ .*

*Proof.*  $\Leftarrow$

Let  $p(\text{Coker}(d)) := \cdots \rightarrow P_{-1} \rightarrow P_0 \xrightarrow{d} P \rightarrow 0 \rightarrow \cdots$ , where  $P_0$  is the projective cover of  $\text{rad } P$ . We claim that the map  $h$  given by

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_{-1} & \longrightarrow & P_0 & \xrightarrow{d} & P & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow d & & \downarrow \text{id} & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & P & \xrightarrow{\text{id}} & P & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

is irreducible in  $\text{Comp}^{-,b}(\mathcal{P})$ .

Suppose the given map factors through a complex  $X \in \text{Comp}^{-,b}(\mathcal{P})$  via maps  $f : p(\text{Coker}(d)) \rightarrow X$  and  $g : X \rightarrow \bar{P}$  such that  $g$  is not a retraction. As  $\text{id}_P = g^1 \circ f^1$ , we have that  $f^1$  is a section and  $g^1$  is a retraction. We have that  $X^1 \cong P \oplus \tilde{P}$  for some projective module  $\tilde{P}$ . If  $\text{Im } g^1 d_X^0 = P$ , then  $g$  is a retraction. Therefore  $\text{Im } g^0 = \text{rad } P$ , as  $\text{Im } d \subset \text{Im } g^1 d_X^0 = \text{Im } g^0$ . Then  $d$  factors through  $g^0$  and  $f^0$ . As  $d$  is irreducible by assumption,  $f^0$  is a section. Therefore  $X^0 = P_0 \oplus P'$  for some projective module  $P'$  and  $g^1 d_X^0(P') = g^0(P') = 0$ . We visualize this in the next diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & P_{-1} & \longrightarrow & P_0 & \xrightarrow{d} & P \longrightarrow 0 \longrightarrow \cdots \\
& & \downarrow & & \downarrow (\text{id}) & & \downarrow (\text{id}) \\
\cdots & \longrightarrow & X^{-1} & \longrightarrow & P_0 \oplus P' & \xrightarrow{\begin{pmatrix} d,0 \\ 0,x \end{pmatrix}} & P \oplus \tilde{P} \longrightarrow 0 \longrightarrow \cdots \\
& & \downarrow & & \downarrow (d,0) & & \downarrow (\text{id},0) \\
\cdots & \longrightarrow & 0 & \longrightarrow & P & \xrightarrow{\text{id}} & P \longrightarrow 0 \longrightarrow \cdots
\end{array}$$

Using the fact that  $\cdots \rightarrow P_{-1} \rightarrow \ker d$  is a minimal projective resolution of  $\ker d$ , we can factorize the map  $g$  through  $s : X \rightarrow p\text{Coker}(d)$  and  $t : p\text{Coker}(d) \rightarrow \bar{P}$  as follows

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & X^{-1} & \longrightarrow & P_0 \oplus P' & \xrightarrow{\begin{pmatrix} d,0 \\ 0,x \end{pmatrix}} & P \oplus \tilde{P} \longrightarrow 0 \longrightarrow \cdots \\
& & \downarrow & & \downarrow (\text{id},0) & & \downarrow (\text{id},0) \\
\cdots & \longrightarrow & P_{-1} & \longrightarrow & P_0 & \xrightarrow{d} & P \longrightarrow 0 \longrightarrow \cdots \\
& & \downarrow & & \downarrow d & & \downarrow \text{id} \\
\cdots & \longrightarrow & 0 & \longrightarrow & P & \xrightarrow{\text{id}} & P \longrightarrow 0 \longrightarrow \cdots
\end{array}$$

So  $s \circ f$  is an isomorphism. Therefore  $f$  is a section.

$\implies$

Conversely, if there is an irreducible map  $p(\text{Coker}(d)) \rightarrow \bar{P}$ , then  $P_0$  is a direct summand of the projective cover of  $\text{rad } P$  and  $d$  is the induced map by 4.2 and 4.3. Let  $r : \tilde{P} \rightarrow \text{rad } P$  be the projective cover of  $\text{rad } P$  and suppose  $\tilde{P} \cong P_0 \oplus P'$  where  $P' \neq 0$  is a projective module and let  $i : P_0 \rightarrow \tilde{P}$  be a section such that  $r \circ i = d$ . Let  $p(\text{Coker}(r)) := \cdots \rightarrow S_{-1} \rightarrow \tilde{P} \xrightarrow{r} P \rightarrow 0 \rightarrow \cdots$ . Then the map of complexes  $p(\text{Coker}(d)) \rightarrow \bar{P}$  induced by the  $p$ -irreducible map  $d$  as in the first part of the proof factors through  $p(\text{Coker}(r))$  as follows:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & P_{-1} & \longrightarrow & P_0 & \xrightarrow{d} & P \longrightarrow 0 \longrightarrow \cdots \\
& & \downarrow & & \downarrow i & & \downarrow \text{id} \\
\cdots & \longrightarrow & S_{-1} & \longrightarrow & \tilde{P} & \xrightarrow{r} & P \longrightarrow 0 \longrightarrow \cdots \\
& & \downarrow & & \downarrow r & & \downarrow \text{id} \\
\cdots & \longrightarrow & 0 & \longrightarrow & P & \xrightarrow{\text{id}} & P \longrightarrow 0 \longrightarrow \cdots
\end{array}$$

The map between the first two rows is not a section and the map between the last two rows is not a retraction. This shows that the map is not irreducible in  $\text{Comp}^{-,b}(\mathcal{P})$ . Therefore  $P_0 \cong \tilde{P}$ .  $\square$



Let  $P_1$  be a summand of the projective cover of  $\text{rad } P$  and  $d : P_1 \rightarrow P$  the induced map. Then  $p(\text{Coker}(d)) \in K^{-,b}(\mathcal{P})$  is isomorphic to the stalk complex  $P/\text{Im } d \in D^b(A)$  and is therefore indecomposable.

**Corollary 4.5.** *Let  $\theta$  be a stable Auslander-Reiten component of  $\Gamma(D^b(A))$ . Then  $l_p$  is not additive if and only if there exists an indecomposable projective module  $P$  such that  $P/\text{rad } P$  has finite projective and finite injective dimension and a shift of the stalk complex  $P/\text{rad } P$  is in  $\theta$ .*

*Proof.* By 3.8 the function  $l_p$  is not additive if and only if there exists an indecomposable complex  $L \in \text{Comp}^b(\mathcal{P})$  and a map  $w : L \rightarrow \nu(L)$  in  $\text{Comp}^b(\mathcal{P})$  that represents an Auslander-Reiten triangle in  $\theta$  such that  $\text{cone}(w)[-1] \in \text{Comp}^b(\mathcal{P})$  contains a contractible summand.

By 3.2 there is a complex  $\bar{P}$  that is a direct summand of  $\text{cone}(w)[-1]$  and an irreducible map  $f : p\nu(L)[-1] \rightarrow \bar{P}$  in  $\text{Comp}^{-,b}(\mathcal{P})$ . By Lemma 4.3,  $\nu(L)[-1]$  is isomorphic in  $D^b(A)$  to  $P/\text{rad } P$  up to shift. As  $\theta$  is a stable Auslander-Reiten component, we have  $p(P/\text{rad } P) \in K^b(\mathcal{P})$  and  $i(P/\text{rad } P) \in K^b(\mathcal{I})$ . Therefore  $P/\text{rad } P$  has finite projective and finite injective dimension.  $\square$

From this corollary it follows that if  $A$  is self-injective, then  $l_p$  is an additive function on the Auslander-Reiten components of  $D^b(A)$ .

We can deduce that Euclidean components always contain the stalk complex of a simple module.

**Theorem 4.6.** *Let  $C$  be a stable component of the Auslander-Reiten quiver of  $D^b(A)$  with tree class an Euclidean diagram. Then  $C$  contains a simple module.*

*Proof.* The function  $l_p$  is subadditive on  $C$ . Suppose  $l_p$  is additive. By [We, 2.4], the function  $l_p$  takes bounded values on  $C$ . This is a contradiction to [S, 4.14]. Therefore  $l_p$  is not additive. By 4.5, this means that  $C$  contains a simple module.  $\square$

As there are only finitely many isomorphism classes of simple modules, we get

**Corollary 4.7.** *The Auslander-Reiten quiver of  $D^b(A)$  contains finitely many Auslander-Reiten components up to shift that have Euclidean tree class. Their number is bound by the number of isomorphism classes of simple  $A$ -modules.*

**Lemma 4.8** (Irreducible maps that do not appear in Auslander-Reiten triangles). *Let  $f : B \rightarrow C$  be an irreducible map in  $D^b(A)$  that does not appear in an Auslander-Reiten triangle. Then  $B, C \notin K^b(\mathcal{P})$  and  $B, C \notin K^b(\mathcal{I})$ .*

*Proof.* By 2.6 it is clear that  $B \notin K^b(\mathcal{I})$  and  $C \notin K^b(\mathcal{P})$ . Let us assume that  $B \in K^b(\mathcal{P})$  and let  $n \in \mathbb{N}$  be minimal such that  $B^n \neq 0$ . Then  $f$  factorizes through  $\sigma^{\geq n-1}(C)$ , where  $C$  is represented as a complex in  $Comp^{-,b}(\mathcal{P})$ . Let  $f = h \circ g$  be this factorization, then  $g$  is not a section, as  $f$  is not a section and  $h$  is not a retraction as  $\sigma^{\geq n-1}(C) \not\cong C$ . This is a contradiction to the fact that  $f$  is irreducible. Therefore  $B \notin K^b(\mathcal{P})$ . Analogously, we can show that  $C \notin K^b(\mathcal{I})$ .  $\square$

An example of such an irreducible map not appearing in the Auslander-Reiten quiver of  $D^b(A)$  is given at the end of [S].

## 5. AUSLANDER-REITEN TRIANGLES OF NAKAYAMA ALGEBRAS

In this section we analyze the Auslander-Reiten quiver of certain Nakayama algebras  $A$  with finite global dimension. The results at the end of this section may help to illustrate results from the previous sections.

Let

$$A_n : 1 \rightarrow 2 \rightarrow \cdots \rightarrow n-1 \rightarrow n$$

and let  $I$  be an ideal of the path algebra of  $kA_n$ . We define  $A := kA_n/I$ , then  $A$  is a Nakayama algebra of finite global dimension. We denote by  $J$  the ideal generated by the paths of length one in  $kA_n$ .

We denote the starting vertex of a path  $b$  by  $s(b)$  and the vertex that  $b$  ends in by  $t(b)$ . With this notation we have  $s(n-1, n) = n$  in the above diagram. Then  $s(b) \geq t(b)$  and  $b_1 b \neq 0$  for two paths  $b_1$  and  $b$  if  $t(b) = s(b_1)$ .

We denote by  $S_i$  the simple modules at the vertex  $i$ , by  $P_i$  the projective indecomposable modules with  $P_i/\text{rad } P_i = S_i$  and by  $I_i$  the injective indecomposable modules with  $\text{soc } I_i = S_i$ . Whenever  $S_j$  occurs in  $I_i$  then it occurs precisely once, therefore  $\dim_k \text{Hom}_A(I_i, I_j) \leq 1$ . So we can fix maps  $d_i^j : I_i \rightarrow I_j$  which are non-zero if and only if  $\dim_k \text{Hom}_A(I_i, I_j) = 1$  such that  $d_i^i = \text{id}$  and  $d_k^j \circ d_i^k = d_i^j$  for all  $1 \leq i \leq k \leq j \leq n$ . When constructing injective resolutions we can always take these maps. Analogously, when we construct minimal projective resolutions we can always take maps  $\nu_A^{-1}(d_i^j)$ .

We investigate the global dimension of Nakayama algebras with quiver  $A_n$ . We call two paths intersecting, if they have at least one arrow in common.

Note that we can always choose a generating set  $B$  of an ideal  $I$  in  $kA_n$  that consists of paths in  $kA_n$ . If the generating set  $B$  is minimal there are no two paths in  $B$  that contain each other. We call a subset of  $B$  intersecting, if every path in that set intersects with at least one other path in the set.

**Lemma 5.1.** *Let  $A = kA_n/I$ . Let  $B$  be a minimal generating set of  $I$  that consists of paths in  $kA_n$ . Then the global dimension of  $A$  is smaller or equal to the maximal cardinality of all intersecting subsets of  $B$  plus one.*

*Proof.* We denote the starting vertex of  $b_i$  by  $s_i := s(b_i)$  and the vertex that  $b_i$  ends in by  $t_i := t(b_i)$ . We assume that paths generating  $B$  are ordered such that  $s_i \leq s_j$  if  $i < j$ . For a fix  $i$  we consider  $\{b_m, \dots, b_i\}$ , the intersecting set of  $B$  that does not contain elements  $b_j$  for  $j > i$  and is maximal with that property.

Suppose first that for all  $m < j < i$  there is an element  $s_w$  for some  $m \leq w \leq i$  with  $s_w \in [t_j, t_{j+1}]$

Then a minimal projective resolution of  $S_{s_i}$  is of the form

$$0 \rightarrow P_{t_m} \rightarrow \dots \rightarrow P_{t_{i-1}} \rightarrow P_{t_i} \rightarrow P_{s_{i-1}} \rightarrow P_{s_i}.$$

Therefore  $S_{s_i}$  has projective dimension  $i - m + 2$ .

Suppose there is some  $j$  with  $m < j < i$  such that the interval  $[t_j, t_{j+1}]$  does not contain any  $s_w$  for all  $m \leq w \leq i$ . Let  $j$  be maximal with this property, then a minimal projective resolution of  $S_{s_i}$  is given by

$$0 \rightarrow P_{t_j} \rightarrow \dots \rightarrow P_{t_{i-1}} \rightarrow P_{t_i} \rightarrow P_{s_{i-1}} \rightarrow P_{s_i}.$$

Therefore  $S_{s_i}$  has projective dimension  $i - j + 2$ . As  $j \geq m + 1$ , we have that the projective dimension is  $\leq i - m + 1$ .

If  $j \neq s_i$  for all  $b_i \in B$ , then a minimal projective resolution of  $S_j$  is given by  $0 \rightarrow P_{j-1} \rightarrow P_j$  and  $S_j$  has therefore projective dimension 1.

We can use the same argument for injective resolutions. This proves the lemma.  $\square$

**Corollary 5.2.** *The algebra  $A$  has global dimension  $n - 1$  if and only if  $I$  is generated by all paths of length 2.*

*Proof.* If  $I$  is generated by all paths of length 2, then we are in the first case of 5.1 and  $S_i$  has projective dimension  $i - 1$  for all  $1 \leq i \leq n$ . Suppose  $A$  has global dimension  $n - 1$ , then  $S_n$  has projective dimension  $n - 1$ . So we are in the first case of 5.1. Therefore a maximal intersecting set of  $B$  is given by

$\{b_1, \dots, b_{n-2}\}$  with  $t_j = j$  and  $s_{n-2} = n$  for all  $1 \leq j \leq n-1$ . Then the  $b_j$  have to be paths of length two. Furthermore  $I$  is generated by the  $b_j$  as any other path in  $A_n$  of length  $> 1$  can be written as a product of the  $b_j$ .  $\square$

By [ASS, 3.5] every indecomposable module  $M$  of  $A$  is isomorphic to  $P_i/\text{rad}^t(P_i)$ . If  $M$  is projective then we choose  $t$  minimal such that  $\text{rad}^t(P_i) = 0$ . Then  $i$  and  $t$  are uniquely determined. We fix  $i$  and  $t$ . Let  $l := l(P_i)$  and  $\bar{l} := l(I_{i-t+1})$ . If  $\text{pdim} M \geq 1$ , then the minimal projective resolution of  $M$  starts with  $\dots \rightarrow P_{i-l} \rightarrow P_{i-t} \rightarrow P_i$ . If  $\text{idim} M \geq 1$  then the minimal injective resolution of  $M$  starts with  $I_{i-t+1} \rightarrow I_{i+1} \rightarrow I_{i+\bar{l}-t+1} \rightarrow \dots$ . We thereby set  $I_k = 0$  and  $P_k = 0$  if  $k \leq 0$ .

We introduce three conditions:

- (1) We have  $d_{i-l}^{i-t+1} = 0$  or  $\text{pdim} M \leq 1$ .
- (2) We have  $d_i^{i+\bar{l}-t+1} = 0$  or  $\text{idim} M \leq 1$ .
- (3) We have  $d_{i-t}^{i+1} = 0$  or  $\text{pdim} M = 0$  or  $\text{idim} M = 0$ .

The next lemma will be used to calculate concrete examples.

**Lemma 5.3.** *Let  $w : M \rightarrow \nu(M)$  define an Auslander-Reiten triangle terminating in  $M$ .*

(a) *cone( $w$ ) has a direct summand isomorphic to  $\nu(\Omega(M))[1]$  if and only if (1) and (3) hold.*

(b) *cone( $w$ ) has a direct summand isomorphic to  $\Omega^{-1}(M)$  if and only if (2) and (3) hold.*

(c) *cone( $w$ ) has a direct summand isomorphic to*

$$\dots \rightarrow 0 \rightarrow I_{i-t+1} \rightarrow I_i \rightarrow 0 \dots$$

*if and only if (1) and (2) hold.*

(d) *cone( $w$ ) decomposes as sum of the indecomposable complexes isomorphic to  $\Omega^{-1}(M)$ ,  $\dots \rightarrow 0 \rightarrow I_{i-t+1} \rightarrow I_i \rightarrow 0 \dots$ , and  $\nu(\Omega(M))[1]$  if and only if (1), (2) and (3) hold.*

(e) *If at most one condition (1)-(3) holds, then cone( $w$ ) is indecomposable.*

*Proof.* We will write  $d$  instead of  $d_i^j$  for an easier presentation. The connecting map  $w$  of the Auslander-Reiten triangle ending in  $M$  can be taken as a map in  $\text{Comp}^{+,b}(T)$

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & I_{i-t+1} & \xrightarrow{d} & I_{i+1} & \xrightarrow{d} & I_{i+\bar{l}-t+1} & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow d & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & I_{i-l} & \xrightarrow{d} & I_{i-t} & \xrightarrow{d} & I_i & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \end{array}$$

Note that if  $\text{pdim} M = 0$ , then  $I_{i-t}$  does not appear and all entries left of  $I_{i-t}$  are zero. If  $\text{pdim} M = 1$ , then all entries left of  $I_{i-t}$  are zero. Analogously if  $\text{idim} M = 0$ , then  $I_{i+1}$  does not appear and all entries right of  $I_{i+1}$  are zero. If  $\text{idim} M = 1$ , then all entries right of  $I_{i+1}$  are zero. Then  $\text{cone}(w)$  is given by the complex

$$\cdots \longrightarrow I_{i-l} \xrightarrow{\begin{pmatrix} d \\ 0 \end{pmatrix}} I_{i-t+1} \oplus I_{i-t} \xrightarrow{\begin{pmatrix} -d, 0 \\ d, d \end{pmatrix}} I_{i+1} \oplus I_i \xrightarrow{\begin{pmatrix} d, 0 \end{pmatrix}} I_{i+l-t+1} \longrightarrow \cdots$$

We determine the direct summands of  $\text{cone}(w)$  in  $\text{Comp}^{+,b}(\mathcal{I})$  as by 3.2 the direct sum decomposition in  $\text{Comp}^{+,b}(\mathcal{I})$  gives the direct sum decomposition in  $D^b(A)$ .

We have a surjective map  $f$  given by

$$\begin{array}{ccccccc} \cdots & \longrightarrow & I_{i-l} & \xrightarrow{\begin{pmatrix} d \\ 0 \end{pmatrix}} & I_{i-t+1} \oplus I_{i-t} & \xrightarrow{\begin{pmatrix} -d, 0 \\ d, d \end{pmatrix}} & I_{i+1} \oplus I_i & \xrightarrow{\begin{pmatrix} d, 0 \end{pmatrix}} & I_{i+l-t+1} & \longrightarrow & \cdots \\ & & \text{id} \downarrow & & (0, \text{id}) \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & I_{i-l} & \xrightarrow{d} & I_{i-t} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Note that up to multiplication with a non-zero scalar, this is the only surjective map between the two complexes. If there is a non-zero map  $g$  from the bottom complex to the top complex such that  $f \circ g = \text{id}$ , then it is of the following form

$$\begin{array}{ccccccc} \cdots & \longrightarrow & I_{i-l} & \xrightarrow{\begin{pmatrix} d \\ 0 \end{pmatrix}} & I_{i-t+1} \oplus I_{i-t} & \xrightarrow{\begin{pmatrix} -d, 0 \\ d, d \end{pmatrix}} & I_{i+1} \oplus I_i & \xrightarrow{\begin{pmatrix} d, 0 \end{pmatrix}} & I_{i+l-t+1} & \longrightarrow & \cdots \\ & & \uparrow \text{id} & & \uparrow \begin{pmatrix} \lambda d \\ \text{id} \end{pmatrix} & & \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & I_{i-l} & \xrightarrow{d} & I_{i-t} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

where  $\lambda \in k$ . We need to determine under which conditions  $g$  is a chain map. As the middle square commutes, we have  $(-\lambda + 1)d_{i-t}^i = 0$  and  $\lambda d_{i-t}^{i+1} = 0$ . As the square on the left hand side commutes, we have  $\lambda d_{i-l}^{i-t+1} = 0$ . If  $\text{pdim} M > 0$ , then  $d_{i-t}^i \neq 0$  as  $\nu_A^{-1}(d_{i-t}^i)$  appears in a minimal projective resolution of  $M$ . Therefore  $\lambda = -1$ . Then  $d_{i-t}^{i+1} = 0$  or  $\text{idim} M = 0$  as in this case  $I_{i+1}$  does not appear in the diagram. This is then equivalent to condition (2). Furthermore we need  $d_{i-l}^{i-t+1} = 0$  or  $\text{pdim} M = 0$  which is equivalent to condition (1). If  $\text{pdim} M = 0$ , then  $I_{i-t}$  and  $I_{i-l}$  do not appear and therefore the diagram commutes.

The bottom row is isomorphic to  $\nu(\Omega(M))[1]$  in  $D^b(A)$ . This is an indecomposable complex, as  $\Omega(M)$  is indecomposable. Therefore  $\text{cone}(w)$  has an indecomposable summand isomorphic to  $\nu(\Omega(M))[1]$ .

We have an injective map  $s$  given by

$$\begin{array}{ccccccc} \cdots & \longrightarrow & I_{i-l} & \xrightarrow{\begin{pmatrix} d \\ 0 \end{pmatrix}} & I_{i-t+1} \oplus I_{i-t} & \xrightarrow{\begin{pmatrix} -d, 0 \\ d, d \end{pmatrix}} & I_{i+1} \oplus I_i & \xrightarrow{(d, 0)} & I_{i+l-t+1} & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \uparrow \text{id} & & \uparrow \text{id} & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & I_{i+1} & \longrightarrow & I_{i+l-t+1} & \longrightarrow & \cdots \end{array}$$

The map  $s$  is up to multiplication with non-zero scalar the only injective map from the bottom row to the top row.

Suppose there is a map  $h$  such that  $h \circ s = \text{id}$ . Then  $h$  is of the form

$$\begin{array}{ccccccc} \cdots & \longrightarrow & I_{i-l} & \xrightarrow{\begin{pmatrix} d \\ 0 \end{pmatrix}} & I_{i-t+1} \oplus I_{i-t} & \xrightarrow{\begin{pmatrix} -d, 0 \\ d, d \end{pmatrix}} & I_{i+1} \oplus I_i & \xrightarrow{(d, 0)} & I_{i+l-t+1} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow (\text{id}, \lambda d) & & \downarrow \text{id} & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & I_{i+1} & \longrightarrow & I_{i+l-t+1} & \longrightarrow & \cdots \end{array}$$

where  $\lambda \in k$ . As in the first case, we need to determine under which conditions  $h$  is a chain map. The middle square commutes if and only if  $(\lambda - 1)d_i^{i+l-t+1} = 0$  and  $\lambda d_{i-t}^{i+1} = 0$ . The square on the left hand side commutes if  $\lambda d_i^{i+l-t+1} = 0$ . If  $\text{idim} M > 0$ , then  $d_i^{i+l-t+1} \neq 0$ . Therefore  $\lambda = 1$ . Then  $d_{i-t}^{i+1} = 0$  or  $\text{idim} M = 1$  and  $d_i^{i+l-t+1} = 0$  or  $\text{pdim} M = 0$ , so condition (2) and (3) hold.

The bottom row is isomorphic to  $\Omega^{-1}(M)$ . As  $A$  is a Nakayama algebra,  $\Omega^{-1}(M)$  is indecomposable. Therefore the complex  $\Omega^{-1}(M)$  is indecomposable.

To prove the third part, we assume that there is a retraction  $x$  from  $\text{cone}(w)$  to the given complex. The map  $x$  and the map  $y$  such that  $y \circ x = \text{id}$  can be taken as

$$\begin{array}{ccccccc} \cdots & \longrightarrow & I_{i-l} & \xrightarrow{\begin{pmatrix} d \\ 0 \end{pmatrix}} & I_{i-t+1} \oplus I_{i-t} & \xrightarrow{\begin{pmatrix} -d, 0 \\ d, d \end{pmatrix}} & I_{i+1} \oplus I_i & \xrightarrow{(d, 0)} & I_{i+l-t+1} & \longrightarrow & \cdots \\ & & \downarrow \uparrow & & \downarrow \uparrow & & \downarrow \uparrow & & \downarrow \uparrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & I_{i-t+1} & \xrightarrow{d} & I_i & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

for some  $\lambda, \delta \in k$ . The square on the left hand side of the map going down commutes if and only if  $\lambda d_{i-l}^{i-t+1} = 0$ . The middle square commutes if and only if  $(\lambda - 1)d_{i-t}^i = 0$ . If  $\text{pdim} M > 0$ , then  $d_{i-t}^i \neq 0$ , so  $\lambda = -1$  and we get that condition (1) is satisfied.

The square on the right hand side of the map going up commutes if and only if  $(\delta + 1)d_{i-t+1}^{i+1} = 0$  holds. The middle square of the map going up commutes if and only if  $\delta d_i^{i+\bar{l}-t+1} = 0$ . If  $\text{idim} M > 0$ , then  $d_{i-t+1}^{i+1} \neq 0$ , so  $\delta = -1$  and  $d_i^{i+\bar{l}-t+1} = 0$  or  $\text{idim} M = 1$ . So condition (2) holds.

Therefore the complex in the bottom row is a direct summand of  $\text{cone}(w)$  if and only if (1) and (2) hold. The complex in the bottom row is indecomposable by 3.4. This proves part (a)-(d).

The complexes  $\sigma^{\geq 1}(\text{cone}(w))$  and  $\sigma^{\leq -1}(\text{cone}(w))$  are indecomposable in  $\text{Comp}^{+,b}(\mathcal{I})$ , as the first one is a minimal injective resolution of an indecomposable module and the second one is  $\nu$  applied to a minimal projective resolution of an indecomposable module. Therefore the retractions presented in the previous three diagrams are the only possibilities for direct summands. So if at most one condition is satisfied, then  $\text{cone}(w)$  has to be indecomposable, which proves part (e). □

We determine the number of predecessors of the simple  $A$ -modules in the Auslander-Reiten quiver of  $D^b(A)$ .

**Lemma 5.4.** *Let  $S_i$  be a simple  $A$ -module and assume that  $S_i$  is not projective and not injective. Then  $S_i$  has two predecessors in  $\Gamma(D^b(A))$  if and only if  $d_{i-1}^{i+1} = 0$ . Otherwise  $S_i$  has only one predecessor.*

*Proof.* With the notation of 5.3, we have that  $t = 1$  and  $\bar{l} = l(I_i)$ . Clearly condition (1) and condition (2) are always satisfied. Therefore  $S_i$  has two predecessors if and only if (3) is satisfied. This is the case if and only if  $d_{i-1}^{i+1} = 0$ . Then  $S_i$  has the two predecessor  $\nu(\text{rad } P_i)$  and  $I_i/S_i[-1]$ . In all other cases  $S_i$  has only one predecessor. □

We compute an example.

**Example 5.5.** *Let  $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4$  and  $I = \langle \alpha\beta\gamma \rangle$ . Then a complete list of indecomposable left  $A$ -modules is given as  $P_1 = S_1 = \text{rad } P_2$ ,  $P_2 = \text{rad } P_3$ ,  $P_3 = I_1$ ,  $M := P_3/P_1 = \text{rad } P_4$ ,  $P_4 = I_2$ ,  $S_2, S_3$ ,  $I_3 = P_4/S_2$  and  $I_4 = S_4$ .*

*As  $P_1$  is simple, its Auslander-Reiten triangle is of the form*

$$P_3[-1] \rightarrow M[-1] \rightarrow P_1 \rightarrow P_3.$$

*As  $I_2 = P_4$  is projective, we have*

$$P_4[-1] \rightarrow \nu^{-1}(i(M)) \rightarrow P_2 \rightarrow P_4.$$

Therefore the Auslander-Reiten triangles of  $P_3$  and  $P_4$  are of the form

$$I_3[-1] \rightarrow \nu(p(M)) \rightarrow P_3 \rightarrow I_3 \text{ and}$$

$$I_4[-1] \rightarrow M \rightarrow P_4 \rightarrow I_4.$$

By 5.3 the module  $M$  satisfies the conditions (1) and (2) and therefore its Auslander-Reiten-triangle is of the form

$$\nu(M)[-1] \rightarrow I_4[-1] \oplus P_3 \oplus S_2 \rightarrow M.$$

The simple modules  $S_2$  and  $S_3$  have only one predecessor by 5.4. Their Auslander-Reiten triangles are of the form

$$S_2 \rightarrow M \rightarrow S_3 \rightarrow S_2[1]$$

and

$$\nu^{-1}(S_3) \rightarrow \nu(M)[-1] \rightarrow S_2 \rightarrow \nu^{-1}(S_3)[1].$$

The  $\tau$ -orbits are the following:

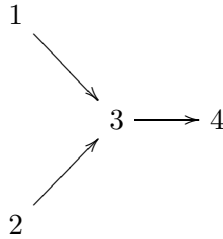
1st orbit:  $P_1, P_3[-1], I_3[-2], P_1[-1]$

2nd orbit:  $P_4, I_4[-1], P_2, P_4[-1]$

3rd orbit:  $M, \nu(M)[-1], \nu^{-1}(M), M[-1]$

4th orbit:  $S_3, S_2, \nu(S_2)[-1], S_3[-1]$ .

So all elements of  $A\text{-mod}$  lie in one component  $\mathbb{Z}[D_4]$  where  $D_4$  is the following quiver:



The number of the vertices of  $D_4$  correspond to the number of the  $\tau$ -orbits and represent their position in the tree of the Auslander-Reiten component. Note also that the shift  $[-1]$  acts on the component as  $\tau^4$ .

By [S, bounded] it is clear that all indecomposable elements of  $K^b(\mathcal{P})$  belong to this component.

We can determine the Auslander-Reiten quiver for some classes of Nakayama algebras.



**Theorem 5.6.** *Let  $A$  be of global dimension  $n - 1$ . Then the Auslander-Reiten quiver is isomorphic to  $\mathbb{Z}[A_n]$ . In particular  $[-1]$  is an involution on the tree class  $A_n$  of  $\Gamma(D^b(A))$ . The algebra  $A$  is derived equivalent to  $kA_n$ .*

*Proof.* Let  $1 < i < n$ , then  $S_i$  is non-projective and non-injective. Furthermore by 5.4 the simple module  $S_i$  has the two predecessors  $\tau(S_{i-1})[1]$  and  $S_{i+1}[-1]$ . We also know by 3.11 that  $S_1$  has the only predecessor  $S_2[-1]$  and  $S_n$  has the only predecessor  $\tau(S_{n-1})[1]$ . So every complex in this component lies in the  $\tau$ -orbit of a stalk complex of a simple module up to shift.

Next we show that all complexes in the component are shift periodic. By direct computation we have that  $\tau^s(S_i)$  is isomorphic to the complex

$$\cdots \rightarrow 0 \rightarrow I_s \rightarrow \cdots \rightarrow I_{i+s-1} \rightarrow 0 \rightarrow \cdots [i - s - 1].$$

Therefore  $\tau^s(S_i)$  has two non-zero homologies except for  $s = n - i + 1$  where  $\tau^{n-i+1}(S_i) = S_{n-i+1}[-n+2i-2]$  and  $s = n+1$  where  $\tau^{n+1}(S_i) = S_i[-2]$ . The  $\tau$ -orbit of  $S_1$  is given by  $\tau^s(S_1) = I_s[-s]$ . Therefore a shift of all projective indecomposable modules and of all injective indecomposable modules are in the orbit of  $S_1$ . Also  $\tau^n(S_1) = S_n[-n]$  and  $\tau(S_n) = S_1[n-2]$ . Therefore  $\tau^{n+1}(S_i) = S_i[-2]$  for all  $1 \leq i \leq n$ . The above computation also shows that the  $\tau$ -orbits of non-isomorphic simple modules are distinct. Therefore a maximal sectional path is given by

$$S_n[-n+1] \longrightarrow \cdots \longrightarrow S_2[-1] \longrightarrow S_1.$$

The Auslander-Reiten component is therefore isomorphic to  $\mathbb{Z}[A_n]$  given as above. As the  $[2]$  shift operates on  $\tau$ -orbits, we can see that all elements in the component are of bounded length. By [S, 4.9] all indecomposable elements are part of the component and by [S, 4.15] the algebra  $A$  is derived equivalent to  $kA_n$ .  $\square$

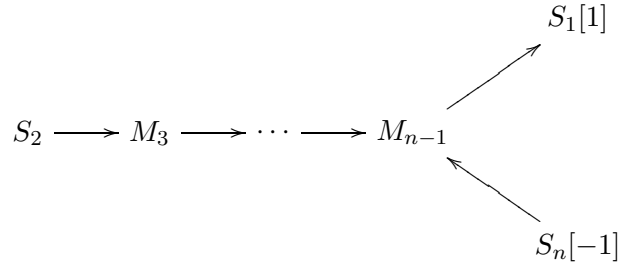
We investigate another class of Nakayama algebras.

**Theorem 5.7.** *Let  $A := kA_n/I$  with  $n \geq 4$  and  $I$  is generated by the path of length  $n - 1$ . Then the Auslander-Reiten quiver of  $D^b(A)$  is isomorphic to  $\mathbb{Z}[D_n]$ . If  $n$  is even, then  $[-1]$  acts as the identity on  $D_n$ . If  $n$  is odd  $[-1]$  acts as the involution on  $D_n$ . Furthermore  $A$  is derived equivalent to  $kD_n$ .*

*Proof.* Let  $1 < i < n$ , then the projective resolution of  $S_i$  is given by  $0 \rightarrow P_{i-1} \rightarrow P_i$  and the injective resolution by  $I_i \rightarrow I_{i+1} \rightarrow 0$ . Therefore  $\tau(S_i) = S_{i-1}$  for  $2 < i < n$ ,  $\tau(S_2) = \cdots \rightarrow 0 \rightarrow I_1 \rightarrow I_2 \rightarrow 0 \rightarrow \cdots$  and  $\tau^2(S_2) =$

$$\cdots \rightarrow 0 \rightarrow I_{i-1} \rightarrow I_{i+1} \rightarrow 0 \rightarrow \cdots$$

Next we investigate the predecessors of modules  $M_s := P_s / \text{rad}^{s-1}(P_s)$  for  $n-1 \geq s \geq 2$ . The projective resolution of  $M_s$  is given by  $0 \rightarrow P_1 \rightarrow P_s$  and the injective resolution of  $M_s$  is given by  $I_2 \rightarrow I_{s+1} \rightarrow 0$ . Therefore (1) and (2) are always satisfied. We have  $d_1^{s+1} = 0$  if and only if  $s = n-1$ . So  $M_s$  has three predecessor if and only if  $s = n-1$  and else it has only two predecessor. By the proof of 5.3, the predecessors of  $M_{n-1}$  are  $S_n[-1]$ ,  $\tau(S_1[1])$  and  $M_{n-2}$ . The predecessors of  $M_s$  for  $2 < s < n-1$  are  $M_{s-1}$  and  $\tau^{-1}(M_{s+1})$ . Therefore a maximal sectional path of the Auslander-Reiten component looks as follows



The component is isomorphic to  $\mathbb{Z}[D_n]$ . Furthermore we know that  $[-2]$  acts on the  $\tau$ -orbit of  $S_i$ ,  $S_n$  and  $S_1$ . Therefore  $[2]$  acts as the identity on all  $\tau$ -orbits of the component. By [S, 4.9] this component is the only Auslander-Reiten component of  $D^b(A)$  and by [S, 4.15] the algebra  $A$  is derived equivalent to  $kA_n$ .  $\square$

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